Discrete spin variables and critical temperature in deterministic models with glassy behavior

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(Received 12 November 1999)

The problem of the existence of a glassy phase transition in deterministic spin models is reconsidered, examining an Ising model with general spin *s* and nontranslationally invariant interaction. The discrete nature of the spin variables is shown to allow the glass state.

PACS number(s): 05.50.+q, 75.50.Lk

I. INTRODUCTION

The spin-glass phase transition in infinite-range systems [like Sherrington- Kirckpatrick (SK) model [1]] has been explained in the framework of replica theory, associating the spin-glass phase with the breaking of symmetry for the replica solution [2]. The replica trick was introduced to treat disorder, which is a key ingredient of the problem. The possibility of describing a glassy transition in the absence of disorder has been reconsidered in recent years by introducing *deterministic* infinite-range spin models with nontranslationally invariant interactions [3–5]. The idea is that deterministic (but highly and irregularly oscillating) couplings among the spins enable to reproduce frustation, yielding a complex landscape for the free energy of the system: one says that disorder is ''selfinduced.'' This scenario has been numerically confirmed in at least one case, the *sine model* [5], through the mapping of the original deterministic system in an appropriate random one. The Hamiltonian of this model is defined by

$$H=-\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j \tag{1}$$

where $\sigma_i=\pm 1,\ i=1,\ldots,N$ are scalar spin variables and J is a symmetric orthogonal $N\times N$ matrix

$$J_{ij}=\frac{2}{\sqrt{2N+1}}\sin\left(\frac{2\pi ij}{2N+1}\right),\quad i,j=1,\ldots,N. \tag{2}$$

It has been observed [6] that the matrix J coincides with the imaginary part of the evolution operator V_A quantizing the elliptic dynamical system given by the unit Hamiltonian sympletic matrix

$$A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

acting over the 2-torus [7]. In this case, a glassy behavior has been detected numerically and also the ground state is known for particular *prime* values of the integer N [5]. While it has been shown using the replica formalism that the system does actually exhibit the glassy behavior of the corresponding random-coupling case, the mean-field equations obtained by resumming the high-temperature expansion do not determine the critical temperature [8]. Another interesting case seems to be the coupling matrix

$$J_{ij}=C_N\frac{1}{\sqrt{N}}\cos\left[\frac{2\pi}{N}(gi^2-ij+gj^2)\right] \tag{4}$$

that corresponds to the real part of the propagator V_B quantizing hyperbolic maps of the form [9]

$$B=\begin{pmatrix} 2g & 1 \\ 4g^2-1 & 2g \end{pmatrix},\quad g\in N. \tag{5}$$

Here, C_N is an arbitrary phase factor, $|C_N|=1$. This time, the critical temperature of a phase transition can be determined by linearization around the largest eigenvalue of J and its value is $T_c\sim 0.8$ [10]. The transition is of glassy type because the mean magnetization is zero for small values of the Edwards-Anderson parameter. We do not know whether the possibility of locate the transition temperature is only a mathematical chance or if there is a deeper physical reason. By the way, we observe that the dynamical system considered in the sine model is periodic and, since J is orthogonal, the eigenvalues are only ± 1. In the second case the dynamical system is chaotic and, at the thermodynamical limit $N\to\infty$ (or equivalently at the classical limit $h\to 0$), the spectrum is equidistributed in $[-1,1]$.

It has been suggested by Parisi *et al.* that, unlike the random case, where the long-ranged spherical model admits a critical temperature [11], the glass transition in deterministic spin models only exist for Ising-like variables. The numerical study of the XY case [12] (where the spin variables are complex numbers of modulo 1) and the analytical one for the spherical model [13] (where the spin variables are allowed to be continuos functions subject to the constraint $\Sigma_{i=1}^{N}\sigma_i=N$) has strenghtened this conjecture, showing that these systems are paramagnetic at all temperatures. In this paper, it is actually shown that it is the discrete nature of the spin variables in deterministic model that generate a phase transition associated with a nontrivial thermodynamical behavior. We are going to generalize the dichotomic Ising case, considering the general case of spin *s*, so that the number of configurations available for each spin is $2s+1$. For the sake of space, we will treat only the ''quadratic'' coupling (4).

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The calculation that will be given is essentially an adaption of the calculation of Ref. [10], part of their work has, however, been simplified.

II. ISING MODEL OF SPIN s

Consider a system of N spins with only one component (the one along the z axis for example) of value s, which can be an integer or a halfinteger. The possible autostates of each spin are labeled by the quantum number s_z which ranges from $-s$ to s in unit steps. The Hamiltonian (normalized to the spin value) is

$$H=-\frac{1}{2s^2}\sum_{i,j=1}^{N} J_{ij}s_z(i)s_z(j)-\frac{1}{s}\sum_{i=1}^{N} s_z(i)h(i). \tag{6}$$

The normalization is such that the maximal interaction between two parallel spins ($\uparrow\uparrow$) remains constant varying the spin value. If $s=1/2$ we recover Eq. (1). The prefactor 1/2 is conventional and is kept to compare with previous results. The partition function [at site-dependent magnetic field $h(i)=0$] is the trace of the Boltzmann factor

$$Z(\beta)=\mathrm{Tr}[\exp(-\beta H)]=\sum_{\{s_z\}}\exp\left[\frac{\beta}{2s^2}\sum_{i,j=1}^{N} J_{ij}s_z(i)s_z(j)\right]. \tag{7}$$

Using standard formulas for gaussian integrations we can rewrite the previous expression as a theory of N fields in zero dimension

$$Z(\beta)=\sum_{\{s_z\}}\frac{1}{\det^{1/2}(2\pi\beta J)}\int_{R^N}dx \times\exp\left[-\frac{1}{2\beta}\sum_{i,j=1}^{N} J_{ij}^{-1}x(i)x(j)+\frac{1}{s}\sum_{i=1}^{N} s_z(i)x(i)\right]. \tag{8}$$

The summation over the s_z is now decoupled and can be carried out; after some algebraic manipulations, we have

$$Z(\beta)=\frac{1}{\det^{1/2}(2\pi\beta J)}\int_{R^N}dx\exp\left(-\frac{1}{2\beta}\sum_{i,j=1}^{N} J_{ij}^{-1}x(i)x(j)+\sum_{i=1}^{N}\log\left\{\frac{\sinh[(1+(1/2s)x(i)]}{\sinh[(1/2s)x(i)]}\right\}\right). \tag{9}$$

To obtain the mean-field equations, we resum the high-temperature expansion for the Gibbs (i.e., magnetization dependent) free energy. We start, as usual, from the Helmholtz free energy $-\beta F(\beta)=\log Z(\beta)$, representing its expansion in diagrammatic way. For the reader's convenience, we recall the fundamental steps.

(1) In the diagrammatic representation we have to consider all the connected diagrams with the propagator βJ_{ij} for any link between two consecutives vertices.

(2) The vertices factors are now generalized for any vertex with m links to the cumulant u_m (i.e., the mth coefficient of Taylor expansion) of $\log\{\sinh[(1+1/2s)x]/\sinh[(1/2s)x]\}$.

(3) Any diagram has to be divided by its order of symmetry. In the thermodynamical limit, due to the properties of quadratic Gauss sums, it can be shown (see Ref. [10] for a rigorous proof) that the set of diagrams contributing in order N are just those of even order $n=2p$ with $p+1$ vertices and p loops. Equivalently these are all the diagrams having two vertices with two links (the extrema) and $p-1$ vertices with four links (all the remaining ones). At every order, Ref. [10] tell us that couplings gives an amount of $N2^{-p}$, the symmetry factor is 2^{p+1} and we only need to calculate the cumulants u_2 and u_4

$$u_2=\frac{1}{3}\frac{s+1}{s}, \tag{10}$$

$$u_4=-\frac{1}{15}\frac{(s+1)(2s^2+2s+1)}{s^3}. \tag{11}$$

Putting everything together, we can perform the summation and obtain the Helmholtz free energy

$$\begin{aligned}-\beta F(\beta)&=N\log(2s+1)+N\sum_{p=1}^{\infty}(u_2)^2(u_4)^{(p-1)}\frac{1}{2^{p+1}}\frac{\beta^{2p}}{2^p}\\&=N\log(2s+1)+\frac{5}{6}\frac{s(s+1)^2\beta^2}{60s^3+(s+1)(2s^2+2s+1)\beta^2}\\&=N\log(2s+1)+NG(\beta).\end{aligned} \tag{12}$$

To determine the Gibbs free energy $\Phi[\beta,m(i)]$, we have to put a magnetic field $h(i)$, repeat the previous expansion and perform the Legendre transform

$$\Phi[\beta,m(i)]\equiv F[\beta,h(i)]+\sum_i h(i)m(i) \tag{13}$$

with $m(i)=-\partial F/\partial h(i)$. The class of nonvanishing diagrams has the same weights as in the $h(i)=0$ case but with an extra factor of $[1-m^2(i_k)]$ for each vertex i_k [14]. The hypothesis of self-averaging Eq. [8], namely $m^2(i)=q\equiv\lim_{N\to\infty}1/N\Sigma_i m^2(i)$ yields the same function $G(\beta)$ in Eq. (12), with β replaced by $\beta(1-q)$. In the final expression for $\Phi[\beta,m(i)]$ we have to put by hand the usual terms given by the entropy of a set of noninteracting spins constrained to have magnetization $m(i)$ and the ''naive'' mean field energy

$$\begin{aligned}-\beta\Phi(\beta,m(i))=&-\sum_i\log\left\{\frac{\sinh[(1+1/2s)\mathcal{L}_s^{-1}(m(i))]}{\sinh[(1/2s)\mathcal{L}_s^{-1}(m(i))]}\right\}\\&-m(i)\mathcal{L}_s^{-1}[m(i)]\\&-\frac{\beta}{2}\sum_{ij}J_{ij}m(i)m(j)-NG[\beta(1-q)],\end{aligned} \tag{14}$$

where

$$\mathcal{L}_s(y)=\left(1+\frac{1}{2s}\right)\coth\left[\left(1+\frac{1}{2s}\right)y\right]-\left(\frac{1}{2s}\right)\coth\left[\left(\frac{1}{2s}\right)y\right] \tag{15}$$

is the so-called Langevin functions [15], which is typical of paramagnetism in nonmetallic solids. Having the expression for the Gibbs free energy, the mean field equations of the model (that are presumably exact because of its infinite range) are given by direct differentiation of Eq. (14). These would be the analogous of TAP equations for SK model [16]. To see whether a ''glass'' phase transition exists we look for solutions of mean-field equations different from the trivial one $q=0$. For T near T_c the magnetizations $m(i)$ and also the eigenvectors corresponding to the largest eigenvalue of J_{ij} are small so we can linearize in $m(i)$. Using

$$\mathcal{L}_s^{-1}[m(i)]=\frac{3s}{s+1}m(i)+\frac{9}{10}\frac{s(1+2s+2s^2)}{(1+s)^3}m^3(i)+o[m^4(i)] \tag{16}$$

the linearized equations read

$$\frac{3s}{s+1}m(i)-\beta m(i)+2\beta G'(\beta)m(i). \tag{17}$$

With some tedious algebra one can rewrite the previous equation as

$$\begin{aligned}0=&\beta^5[(1+s)^3(1+2s+2s^2)^2]-\beta^4[3s(1+s)^2(1+2s\\&+2s^2)^2]+\beta^3[120s^3(1+s)^2(1+2s+2s^2)]\\&-\beta^2[40s^4(1+s)(14+28s+23s^2)]\\&+\beta[3600s^6](1+s)-10800s^7.\end{aligned} \tag{18}$$

The critical temperature T_c is given by the zeros of Eq. (18). For fixed value of s we numerically solved it; in Fig. 1 we plot the critical inverse temperature β_c versus the spin value s. The numerical fit is consistent with the following law:

$$\beta_c\sim\frac{10}{3}-\frac{3}{s+1}. \tag{19}$$

The inverse critical temperature grows with an inverse power law for increasing s, approaching a constant value for $s\to\infty$.

We have confirmed in this way that discreteness of spin variable allow the existence of a phase transition in deterministic models. Its nature is ''glassy'' (in the sense that the pure magnetization states at zero temperature are neither ferromagnetic nor antiferromagnetic) by the same argument of Ref. [10]: at $T=0$ the averaged magnetization is zero because it is proved that

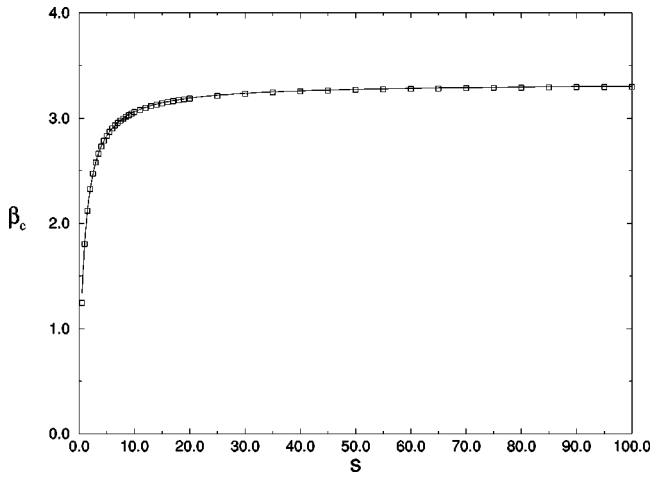

FIG. 1. Plot of the inverse critical temperature β_c versus the spin value s. The solid line is the curve $y=\frac{10}{3}-3/s+1$.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mu_i=0$$

if μ_i are the components of any normalized eigenvector of the matrix J corresponding to the eigenvalue 1. Even though the ground state is degenerate, none of the magnetization states generates long-range order. One would ask if the discreteness of spin variable is a necessary condition to a glass state exist. To obtain this conclusion one would have to show that there is no transition in models with continuous symmetry (like the XY or Heisenberg models). This point goes beyond the aims of this paper and will be discussed elsewhere.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor S. Graffi for suggesting the problem and for a critical reading of the manuscript. We are also indebted to A. Gruppuso, S. Isola, and F. Unguendoli for useful discussions and comments.

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